Pergamon

Int. J. Heat Mass Transfer. Vol. 41, No. 10, pp. 1327–1337, 1998

Printed in Great Britain
0017–9310/98 $19.00+0.00

PII: S0017–9310(97)00126–9

Weakly non-linear Marangoni instability in the presence of a magnetic field

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(*Received* 12 *July* 1996 *and in final form* 25 *April* 1997)

Abstract—Weakly non-linear Marangoni convection in a horizontal layer of an electrically conducting fluid heated from below and submitted to a uniform vertical magnetic field is investigated. The influence of the strength of the magnetic field on the nature of the convective patterns and their stability is analyzed by using the amplitude method. To examine the role of the thermal properties of the lower rigid boundaries, two different situations are, respectively, treated, namely a perfectly heat conducting and an adiabatically isolated wall. The presence of a subcritical region whose area generally decreases with the magnetic field is displayed. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

Convective instability in thin fluid layers heated from below and driven by surface-tension variations has been the subject of numerous works since Bénard's remarkable experiments. When the single motor of instability is the variation of surface tension with temperature, one speaks about the Marangoni effect; the first theoretical analysis of Marangoni convection was that of Pearson [1]. When only buoyancy is responsible for the occurrence of motion, the effect is usually referred to as Rayleigh–Bénard effect. The latter is widely discussed in Chandrasekhar's celebrated book [2], at least the linear problem.

Convective flows are of practical importance in several domains of application, like liquid bridges, pool boiling, motion through porous media, crystal growth from a melt, thermal transfer problems, etc., but it is often desirable to delay the onset of convection; this can, in particular, be achieved by the application of a magnetic field. The interaction between magnetic field and convection is directly observed on the sun; due to high temperature fields, the gases are ionized and a strong magnetic field is generated which inhibits normal convection. It is well recognized [3] that when a magnetic field is imposed on an electrically conducting fluid, the fluid motion is slowed down because of the interaction between the induced electrical current and the external magnetic field. This kind of interaction is frequently met in geophysics, astrophysics and engineering.

Thermal convection in presence of a magnetic field has been extensively investigated during the last decenny. However, most studies concern buoyancy driven convection to which several linear [1] and non-linear analyses [4–9] where dedicated. Only a few papers deal with the Marangoni instability [10–13], but all of them are restricted to a linear approach. From the previous works, we particularly notice the following properties:

(1) The effect of the magnetic field is negligibly small when the Chandrasekhar number Q, which is a dimensionless number measuring the effect of the magnetic field, is smaller than unity; in contrast, the influence of the magnetic field becomes relevant when Q becomes greater than 10 [10].
(2) The ratio between thermal and magnetic diffusivity κ/r_e plays an important role in magneto-convection; indeed, if $\kappa/r_e \ll 1$ convection is steady, while for $\kappa/r_e \gg 1$ convection sets in as an oscillatory instability [1, 11].
(3) In the specific problem of convective motion at the surface of the sun, the Prandtl number Pr is less than unity and the magnetic field is very strong [14]. However, for most liquids used in experiments on earth, the Prandtl numbers are rather large as they range from 100 to 1000. It was shown from Scanlon and Segel's non-linear analysis [15] that the influence of the Prandtl number on convection problem is of order of the ratio $Pr/(1+Pr)$, from which follows that the approximation of an infinite Prandtl number is reasonable as soon as $Pr > 5$. The systems we have in mind in this work are viscous magnetic layers, suspensions of small magnetic particles in a fluid carrier. Typically for the magnetic fluid EMG widely used in experiments on earth, one has $Pr = 100$.

The purpose of the present work is to propose a

NOMENCLATURE

a	dimensionless wave number
d	thickness of the layer
$\bar{D}$ ($= d/dz$)	non-dimensional z-space derivative
$\mathbf{H}_0$	strength of the applied uniform magnetic field
$\mathbf{H}(H_x, H_y, H_z)$	magnetic field
h_T	heat transfer coefficient
h	Biot number, $h_T d\lambda^{-1}$
ℓ	horizontal extent
L	linear growth disturbance coefficient
$\bar{L}$	linear operator
M	linear operator
Ma	Marangoni number $(-\partial_T\xi)(\kappa\rho\nu)^{-1}\beta_T d^2$
N, N_1, N_2	non-linear operators
p	pressure
P_m	magnetic Prandtl number, νr_e^{-1}
Pr	Prandtl number, $\nu\kappa^{-1}$
Q	Chandrasekhar number, $\eta(H_0 d)^2(4\pi\rho\nu r_e)^{-1}$
r_e	electrical resistivity
T	temperature
T_1	dimensionless linear temperature solution
T_i	dimensionless second-order temperature solutions
T_0	temperature at the lower surface
T_u	temperature at the upper surface
$\mathbf{u}(u, v, w)$	velocity field
z, y, z	space coordinates
Y, Z	time-dependent perturbation amplitudes.

Greek symbols

α, β	constants
β_T	vertical temperature gradient $(T_0 - T_u)d^{-1}$
γ	coefficients of quadratic terms in amplitude equations
∂_T	partial temperature derivative operator
∂_t	partial time derivative
$\partial_x, \partial_y, \partial_z$	partial space derivative with respect to x, y and z
$\partial_{xx}, \partial_{yy}, \partial_{zz}$	second-order space derivatives with respect to x, y and z
$\varepsilon_c, \varepsilon_1, \varepsilon_2$	coefficients delimiting the stability domain of the selected cell pattern
ε	distance to the threshold $(Ma - Ma^c)(Ma^c)^{-1}$
κ	thermal diffusivity
λ	thermal conductivity
η	magnetic permeability
ν	kinematic viscosity
$\nabla(\partial_x, \partial_y, \partial_z)$	nabla operator
∇_1^2	horizontal laplacian operator, $\partial_{xx} + \partial_{yy}$
ξ	surface tension
ρ	density
ϕ	selected planform function
ϕ_i	auxiliary functions ($i = 1, \ldots, 4$).

Subscripts

r	reference solution
$)_z$	z-fixed coordinate.

Superscripts

*	adjoint quantity
c	critical value
(n)	the order of solution ($n = 1, 2$).

weakly non-linear analysis of Marangoni instability problem in a thin electrically heat conducting layer, of infinite Prandtl number and submitted to a normal magnetic field. In particular we wish to emphasize the role of the magnetic field and the nature of the thermal properties at the lower boundary on the nature of the convective pattern observed beyond the critical instability threshold.

Experiments indicate that the preferred convective structures are either rolls, hexagons or hybrid structures. To solve the problem we use the procedure proposed by Scanlon and Segel [15], and revisited by Bragard and Lebon [16], which consists essentially of expressing the relevant fields in terms of time-dependent amplitudes.

2. THE MATHEMATICAL MODEL

Consider an electrically conducting horizontal fluid layer of infinite horizontal extent and thickness d, bounded below by a rigid plane, which is either perfectly heat conducting or adiabatically insulated; the upper surface is free and open to air, it is heat conducting, undeformable, but subject to a surface tension $\xi(T)$ decreasing linearly with the temperature T:

$$\xi(T) = \xi(T_0) + \frac{\partial\xi}{\partial T}(T - T_0) \quad \left(\frac{\partial\xi}{\partial T} < 0\right) \tag{1}$$

T_0 is a reference temperature, say the temperature T_0 at the lower boundary. The layer, assumed to be incompressible, is heated from below and submitted to a homogeneous vertical magnetic field $\mathbf{H}_0$. A reference frame xyz is attached at the lower plate with the z-direction normal to the fluid and orientated in the direction of the magnetic field $\mathbf{H}_0$. In the reference state the fluid is at rest and heat propagates only by conduction; the corresponding velocity $\mathbf{v}(u, v, w)$ and temperature T fields are given by

$$v_r = 0, \quad T_r = T_0 - \beta_T z \tag{2}$$

β_T stands for $(T_0 - T_u)/d$ where T_u is the temperature at the upper boundary. Within Boussinesq's approximation, the governing equations for the perturbations of the reference state characterized by a uniform vertical magnetic field and absence of motion are [3]:

$$\nabla \cdot \mathbf{v} = 0 \tag{3}$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{\eta}{4\pi\rho}[(\mathbf{H} \cdot \nabla)\mathbf{H} + H_0 \mathbf{H}_{,z}] = -\nabla\left[\frac{p}{\rho} + \frac{\eta|\mathbf{H}|^2}{8\pi\rho}\right] + \nu\nabla^2\mathbf{v} \tag{4}$$

$$\partial_t T + (\mathbf{v} \cdot \nabla)T = \kappa\nabla^2 T + \beta_T w \tag{5}$$

$$\partial_t \mathbf{H} + (\mathbf{v} \cdot \nabla)\mathbf{H} = (\mathbf{H} \cdot \nabla)\mathbf{v} + H_0 \mathbf{v}_{,z} + r_e \nabla^2 \mathbf{H} \tag{6}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{7}$$

In equations (3)–(7), we have introduced the following notation: $\mathbf{H}(H_x, H_y, H_z)$ is the magnetic field in the fluid, p the pressure, η the magnetic permeability, r_e the electrical resistivity, ν the kinematic viscosity, ρ the density, κ the heat diffusivity; we recall that gravity is neglected everywhere. In the energy balance, we have omitted not only the viscous heating, but also the Joule heating. Applying twice the curl operator on equation (4) and using the constraints (3) and (7), one obtains the following relations for the x, y, z-components written in dimensionless form; distance, time, temperature and magnetic field have been scaled by d, d^2/κ, βd and $H_0\kappa/r_e$, respectively:

$$Pr^{-1}[\partial_t \nabla^2 w + \nabla_1^2(\mathbf{v} \cdot \nabla w) - QP_m \nabla_1^2(\mathbf{H} \cdot \nabla H_z) - \partial_{zx}(\mathbf{v} \cdot \nabla u) - \partial_{zy}(\mathbf{v} \cdot \nabla u) + [QP_m\, \partial_{zx}(\mathbf{H} \cdot \nabla H_x) + \partial_{zy}(\mathbf{H} \cdot \nabla H_y)] = \nabla^4 w + Q\,\partial_z \nabla^2 H_z \tag{8}$$

$$Pr^{-1}[\partial_t(\nabla_1^2 u + \partial_{xz} w) + \partial_{yy}(\mathbf{v} \cdot \nabla u) + \partial_{xy}(\mathbf{v} \cdot \nabla v) - QP_m\, \partial_{yy}(\mathbf{h} \cdot \nabla H_x)] - Q\,\partial_z(\nabla_1^2 H_x + \partial_{xz} H_z) = \nabla^2(\nabla_1^2 u + \partial_{xz} w) \tag{9}$$

$$Pr^{-1}[\partial_t(\nabla_1^2 v + \partial_{yz} w) + \partial_{xx}(\mathbf{v} \cdot \nabla v) - \partial_{xy}(\mathbf{v} \cdot \nabla u) - QP_m\, \partial_{xx}(\mathbf{h} \cdot \nabla H_y)] - Q\,\partial_z(\nabla_1^2 H_y + \partial_{yz} H_z) = \nabla^2(\nabla_1^2 v + \partial_{yz} w). \tag{10}$$

The z-component of equation (6) is given by

$$Pr^{-1} P_m[\partial_t H_z + (\mathbf{v} \cdot \nabla H_z) - (\mathbf{H} \cdot \nabla w)] = \partial_z w + \nabla^2 H_z. \tag{11}$$

Expressions of the horizontal components H_x and H_y of the magnetic field are obtained by applying twice the curl operator on equation (6) and leads to:

$$Pr^{-1} P_m[\partial_t(\nabla_1^2 H_x + \partial_{zx} H_z) + \partial_{yy}(\mathbf{v} \cdot \nabla H_x) - \partial_{xy}(\mathbf{v} \cdot \nabla H_y) + \partial_{xy}(\mathbf{H} \cdot \nabla v) - \partial_{yy}(\mathbf{H} \cdot \nabla u)] = \partial_z(\nabla_1^2 u + \partial_{xz} w) + \nabla^2(\nabla_1^2 H_x + \partial_{zx} H_z) \tag{12}$$

$$Pr^{-1} P_m[\partial_t(\nabla_1^2 H_y + \partial_{zy} H_z) + \partial_{xx}(\mathbf{v} \cdot \nabla H_y) - \partial_{xy}(\mathbf{v} \cdot \nabla H_x) + \partial_{xy}(\mathbf{H} \cdot \nabla u) - \partial_{xx}(\mathbf{H} \cdot \nabla v)] = \partial_z(\nabla_1^2 v + \partial_{yz} w) + \nabla^2(\nabla_1^2 H_y + \partial_{zy} H_z). \tag{13}$$

The Prandtl number Pr, the magnetic Prandtl P_m and Chandrasekhar number Q are defined, respectively, as:

$$Pr = \frac{\nu}{\kappa}, \quad P_m = \frac{\nu}{r_e}, \quad Q = \frac{\eta H_0^2 d^2}{4\pi\rho r_e \nu}.$$

It is important to distinguish between the magnetic Prandtl number, P_m and the viscous Prandtl number, Pr. Here we take $P_m < Pr$ to guarantee the exchange of stability. Of course, this condition is automatically satisfied by taking $Pr = \infty$, and this is supplementary motivation for setting $Pr = \infty$. Since it is assumed that $Pr = \infty$, the only non-linear terms are those appearing in the energy equation (5). Expressions (8)–(13) simplify then as follows:

$$\nabla^4 w - Q\,\partial_{zz} w = 0 \tag{14}$$

$$\nabla^2 T + w = \partial_t T + (\mathbf{v} \cdot \nabla T) \tag{15}$$

$$\nabla_1^2 u = -\partial_{xz} w \tag{16}$$

$$\nabla_1^2 v = -\partial_{yz} w \tag{17}$$

$$\nabla^2 H_z = -\partial_z w \tag{18}$$

$$\nabla_1^2 H_x = -\partial_{xz} H_z \tag{19}$$

$$\nabla_1^2 H_y = -\partial_{yz} H_z. \tag{20}$$

It is observed that the problem is characterized by a separation of variables so that one can solve the two first equations (14)–(15) independently of the remaining equations (16)–(20). The next step consists in solving equations (14)–(15) associated with the following boundary conditions:

$$\text{at } z = 0: W = \partial_z w = 0 \tag{21}$$

$$T = 0 \text{ (conducting case)}, \quad \partial_z T = 0 \text{ (insulating case)} \tag{22}$$

$$\text{at } z = 1: \partial_{zz} w - Ma\nabla_1^2 T = 0 \tag{23}$$

$$\partial_z T + hT = 0 \tag{24}$$

$$w = 0 \tag{25}$$

wherein Ma and h stand for the dimensionless Marangoni and Biot numbers defined, respectively, by

$$Ma = \left(-\frac{\partial \xi}{\partial T}\right)\frac{\beta_T d^2}{\rho \nu \kappa}, \quad h = \frac{h_T d}{\lambda} \tag{26}$$

with h_T the heat transfer coefficient through the upper surface and λ the heat conductivity of the fluid. Relation (23) expresses the equilibrium between the viscous forces and the surface tension efforts. Equation (24) describes the heat transfer at the upper boundary, while equation (25) states that the upper surface remains flat.

3. WEAKLY NON-LINEAR SOLUTION

The non-linear equations (14) and (15) and the associated boundary conditions (21)–(25) are solved by means of an iterative process. With Scanlon and Segel [15], we introduce the following differential operators:

$$\bar{L} = \begin{bmatrix} \nabla^4 - Q\,\partial_{zz} & 0 & 0 \\ 1 & \nabla^2 & 0 \\ \partial_{zz})_{z=1} & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \partial_t + \mathbf{v}\cdot\nabla & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \nabla_1^2 \end{bmatrix}$$

in terms of which equations (14)–(25) take the symbolic form

$$\bar{L}(\mathbf{u}) = N(\mathbf{u}) + MaM(\mathbf{u}) \tag{27}$$

where $\mathbf{u}$ is the perturbation vector with components

$$\mathbf{u} = [w(x,y,z,t), T(x,y,z,t), T(x,y,1,t)]. \tag{28}$$

For further purpose, it is also necessary to define the scalar product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \lim_{\ell \to \infty} \frac{1}{4\ell^2} \int_{-\ell}^{\ell} \int_{-\ell}^{\ell} \mathrm{d}x\,\mathrm{d}y [\mathrm{d}z(a_1 b_1 + a_2 b_2) + (a_3 b_3)_{z=1}] \tag{29}$$

where ℓ represents the horizontal extent. In view of the iterative procedure, expression (27) will be written as

$$[\bar{L} - \mathrm{Ma^c M}]\mathbf{u}^{(n)} = [N + (Ma - Ma^c)M]\mathbf{u}^{(n-1)} \tag{30}$$

where the right-hand side contains only small terms, and where Ma^c is the critical Marangoni number obtained from the linear theory. In view of future developments, we define the distance from the threshold by $\varepsilon = (Ma - Ma^c)/Ma^c$, assumed to remain small in order to ensure convergence.

3.1. *The linear solution*

The first step consists in solving the linear problem corresponding to $n = 1$ and $\mathbf{u}^{(0)} = 0$. The solution $\mathbf{u}^{(1)}$ can be written as

$$\mathbf{u}^{(1)} = [w_1(z), T_1(z), T_1(1)]\phi(x,y,t) \tag{31}$$

where the form function $\phi(x,y,t)$ is solution of the Helmholtz equation

$$\nabla_1^2 \phi + a^2 \phi = 0 \tag{32}$$

with a the dimensionless wavenumber. The functions $w_1(z)$ and $T_1(z)$ are solutions of the differential equations

$$[(\bar{D}^2 - a^2)^2 - Q\bar{D}^2]w_1 = 0 \tag{33}$$

$$(\bar{D}^2 - a^2)T_1 + w_1 = 0 \tag{34}$$

associated to the following boundary conditions:

$$\text{at } z = 0: w_1 = \bar{D}w_1 = 0 \tag{35}$$

$$T_1 = 0 \text{ (conducting case)} \tag{36}$$

$$\bar{D}T_1 = 0 \text{ (insulating case)} \tag{37}$$

$$\text{at } z = 1: w_1 = \bar{D}T_1 + hT_1 = 0 \tag{38a, b}$$

$$\bar{D}^2 w_1 + a^2 MaT_1 = 0 \tag{39}$$

where $\bar{D}$ stands for d/dz. It is easily checked that the following expressions for $w_1(z)$ and $T_1(z)$ verify the boundary conditions (35)–(38):

$$w_1(z) = A(\cosh \alpha z + B \sinh \alpha z + C \cosh \beta z + D \sinh \beta z) \tag{40}$$

$$T_1(z) = A\left(E \sinh az + F \cosh az + \frac{\cosh \alpha z + B \sinh \alpha z}{a^2 - \alpha^2} + \frac{C \cosh \beta z + D \sinh \beta z}{a^2 - \beta^2}\right) \tag{41}$$

A is an arbitrary constant while the quantities α, β, B, C, D, E and F depend on h, Q and a. The unknown quantities B, C and D appearing in expression (40) of $w_1(z)$ are obtained by imposing $w_1(z)$ to satisfy the boundary conditions (35) and (38a); it is found that

$$B = \frac{\cosh \beta - \cosh \alpha}{\sinh \alpha - \frac{\alpha}{\beta} \sinh \beta}; \quad C = -1; \quad D = -\frac{\alpha}{\beta} B \tag{42}$$

wherein

$$\alpha = \frac{Q^{1/2} + \Delta^{1/2}}{2}; \quad \beta = \frac{Q^{1/2} - \Delta^{1/2}}{2}; \quad \Delta = Q + 4a^2. \tag{43}$$

The quantities E and F in expression (41) for $T_1(z)$ are obtained from the boundary conditions (36), (37) and (38b): for a perfect heat conductor at $z = 0$, one has

$$F = \frac{1}{a^2-\beta^2} - \frac{1}{a^2-\alpha^2} \tag{44}$$

$$E = -\frac{1}{(a\cosh a + h\sinh a)}\left[F(a\sinh a + h\cosh a) + \alpha\frac{\sinh\alpha + B\cosh\alpha}{a^2-\alpha^2} + \beta\frac{C\sinh\beta + D\cosh\beta}{a^2-\beta^2} + h\left(\frac{\cosh\alpha + B\sinh\alpha}{a^2-\alpha^2} + \frac{C\cosh\beta + D\sinh\beta}{a^2-\beta^2}\right)\right] \tag{45}$$

and, for an insulating wall, one obtains

$$E = -\frac{\alpha}{a}B\left(\frac{1}{a^2-\alpha^2} - \frac{1}{a^2-\beta^2}\right) \tag{46}$$

$$F = \frac{-1}{(a\sinh a + h\cosh a)}\left[E(a\cosh a + h\sinh a) + \frac{\alpha(\sinh\alpha + B\cosh\alpha)}{a^2-\alpha^2} + \beta\frac{C\sinh\beta + D\cosh\beta}{a^2-\beta^2} + h\left(\frac{\cosh\alpha + B\sinh\alpha}{a^2-\alpha^2} + \frac{C\cosh\beta + D\sinh\beta}{a^2-\beta^2}\right)\right]. \tag{47}$$

Condition (39) yields the expression of the Marangoni number in terms of the wavenumber, the Biot and Chandrasekhar numbers for both conducting and insulating cases: it is found that

$$Ma = \frac{-\alpha^2(\cosh\alpha + B\sinh\alpha) - \beta^2(C\cosh\beta + D\sinh\beta)}{a^2\left(E\sinh a + F\cosh a + \dfrac{\cosh\alpha + B\sinh\alpha}{a^2-\alpha^2} + \dfrac{C\cosh\beta + D\sinh\beta}{a^2-\beta^2}\right)}. \tag{48}$$

The minimum of the neutral stability curve, $Ma(a)$ vs a (for fixed values of Q and h) yields the corresponding critical Marangoni number Ma^c and the critical wavenumber a^c. The critical Marangoni and wavenumbers for different values of Q and h in both the conducting and insulating cases are reported in Figs. 1(a) and (b).

It is noticed that by increasing Q or h, the system becomes more stable as confirmed experimentally. It is known that the linear theory does not predict the shape of the convective cells. Experimentally it is observed that these cells usually take the form of either hexagons or rolls or hybrid structures. To determine the geometry of the pattern, one needs to proceed to the second-order solution of the problem.

3.2. *The second-order solution*

The form function ϕ is selected in order to match the experimental observations; it will be written in terms of two spatial modes of the same wavenumber a with two unknown amplitudes $Z(t)$ and $Y(t)$ depending on the time: explicitly one has

$$\phi(x,y,t) = Z(t)\cos(ay) + Y(t)\cos(m_1 ax)\cos(m_2 ay) \tag{49}$$

with $m_1 = \sqrt{3/2}$ and $m_2 = 1/2$. For $Y = 0$ and Z = constant, the solution (49) consists of rolls, while for $Y = \pm 2Z$, it describes hexagonal cells. The second-order solution is obtained by integrating the inhomogeneous differential equations (30) for $n = 2$. Existence of non-trivial solutions is ensured by the Fredholm condition [15], stating that

$$\langle \mathbf{u}^*, [N + (Ma - Ma^c)M]\mathbf{u}^{(1)}\rangle = 0 \tag{50}$$

where

$$\mathbf{u}^* = [w^*(x,y,z,t), T^*(x,y,z,t), T^*(x,y,1,t)] \tag{51}$$

is the solution of the linear adjoint problem, which is given in Appendix A. The orthogonality condition imposes that the amplitudes $Y(t)$ and $Z(t)$ satisfy the ordinary differential equations

$$\dot{Y} = L\varepsilon Y + \gamma YZ; \quad \dot{Z} = L\varepsilon Z + \frac{\gamma}{4}Y^2 \tag{52}$$

wherein a dot means derivation with respect to time, the coefficients L and γ are easy to determine and depend generally on the Biot and the Chandrasekhar numbers h and Q.

The second-order solution

$$\mathbf{u}^{(2)} = [w^{(2)}(x,y,z,t), T^{(2)}(x,y,z,t), T^{(2)}(x,y,1,t)] \tag{53}$$

obeys the equations

$$\nabla^2 T^{(2)} + w^{(2)} = T_1\dot{\phi} + T_1\bar{D}w_1\left(\phi_1 + \frac{\phi_2}{2} - \frac{\phi_3}{2} - \phi_4\right) + w_1\bar{D}T_1(\phi_1+\phi_2+\phi_3+\phi_4) \tag{54}$$

$$\nabla^4 w^{(2)} - Q\bar{D}^2 w^{(2)} = 0 \tag{55}$$

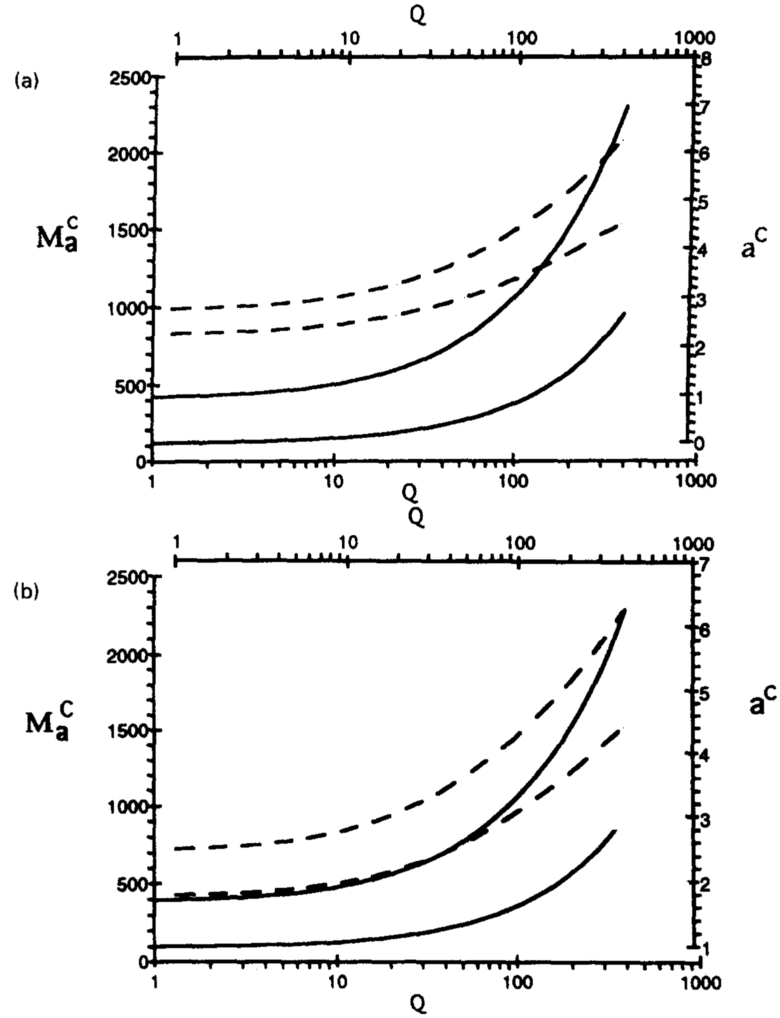

Fig. 1. (a) Critical Marangoni number (solid lines) and critical wavenumber (dashed lines) vs Chandrasekhar number in the conducting case. The values of the Biot number are 1, 10; the lower curve represents the smallest Biot number; (b) critical Marangoni number (solid lines) and critical wavenumber (dashed lines) vs Chandrasekhar number in the insulating case. The values of the Biot number are 1, 10; the lower curve represents the smallest Biot number.

$$\bar{D}^2 w^{(2)} - Ma^c \nabla_1^2 T^{(2)} = \varepsilon Ma^c \nabla_1^2 T^{(1)} \tag{56}$$

associated to the boundary conditions:

$$\text{at } z = 0: w^{(2)} = \bar{D}w^{(2)} = 0 \tag{57}$$

$$T^{(2)} = 0 \text{ (conducting case)} \tag{58}$$

$$\bar{D}T^{(2)} = 0 \text{ (insulating case)} \tag{59}$$

$$\text{at } z = 1: w^{(2)} = \bar{D}T^{(2)} + hT^{(2)} = 0. \tag{60}$$

The functions ϕ_i ($i = 1, 2, 3, 4$) appearing in equation (54) are defined by

$$\phi_1 = \frac{Z^2}{2} + \frac{Y^2}{4} \tag{61}$$

$$\phi_2 = \frac{Y^2}{4}\cos(ay) + YZ\cos(m_1 ax)\cos(m_2 ay) \tag{62}$$

$$\phi_3 = \frac{Y^2}{4}\cos(2m_1 ax) + YZ\cos(m_1 ax)\cos(3m_2 ay) \tag{63}$$

$$\phi_4 = \frac{Z^2}{2}\cos(2ay) + \frac{Y^2}{4}\cos(2m_1 ax)\cos(ay). \tag{64}$$

The solution of equations (54)–(60) may be written in the form

$$w^{(2)} = w_3(z)\phi_3 + w_4(z)\phi_4 \tag{65}$$

$$T^{(2)} = K(z)\dot{\phi} + T_1(z)\phi_1 + T_2(z)\phi_2 + T_3(z)\phi_3 + T_4(z)\phi_4 \tag{66}$$

where the unknown functions w_3, w_4, K, T_i ($i = 1, 2, 3, 4$) are made explicit in Appendix B. The application of the Fredholm condition (50) on the set (54)–(60) results in

$$\int_v T^* \left(\frac{\partial T^{(1)}}{\partial t} + N_1 T^{(1)} + N_1 T^{(2)} + N_2 T^{(1)} \right) dv + (Ma - Ma^c) \int_{S(z=1)} \frac{\partial w^*}{\partial z} \nabla_1^2 T^{(1)} \, dS = 0 \quad (67)$$

wherein $N_1 = \mathbf{u}^{(1)} \cdot \nabla$, $N_2 = \mathbf{u}^{(2)} \cdot \nabla$ while v is the volume of a cell bounded by a surface S. In (67) only two integrals proportional to $\cos^2(ay)$ and $\cos^2(m_1 ax)$ $\cos^2(m_2 ay)$ are not zero and they give rise to the third-order amplitude equations

$$\dot{Y} = L\varepsilon Y + \gamma YZ + PYZ^2 + RY^3 \quad (68)$$

$$\dot{Z} = L\varepsilon Z + \frac{\gamma}{4} Y^2 + R_1 Z^3 + \frac{P}{2} Y^2 Z. \quad (69)$$

Equations (68) and (69) are formally the same as those found by Scanlon and Segel [15] and Bragard and Lebon [16, 17] with the difference that here the coefficients L, γ, R_1, P and $R = (P_1 + R)/4$ depend on the Chandrasekhar number besides Biot's number. In deriving the relations (68) and (69), it is assumed that the function ϕ is proportional to $\gamma \phi_2$ [18]. This assumption allows to eliminate the second order derivatives $\ddot{Y}$ and $\ddot{Z}$ and does not alter the stability properties of the steady solutions of (68) and (69), as shown by Segel and Stuart [18].

The next problem consists in determining the steady solutions of (68) and (69) and to discuss their stability with respect to small disturbances. Each steady solution represents either a state of conduction without convection or a convective pattern, e.g. rolls, hexagons or hybrid cells. Their stability will depend on the value of the parameter ε measuring the deviation from the critical Marangoni number.

3.3. *Stability of the steady solutions*

In this section, we first examine the behaviour of the coefficients appearing in the amplitude equations (68) and (69) as a function of the Chandrasekhar number Q. The factor L expresses the linear growth of the disturbance and is found to increase with the increasing values of the Chandrasekhar number. The coefficients (P, R, R_1) of the third order terms of the amplitude equations (68) and (69) are found to be negative, whatever the values of Q and h, while the coefficient γ of the quadratic terms remains positive. If ε is negative (subcritical domain), the disturbance decreases more rapidly with time in the conducting case than in the insulating case for the same Biot number. For positive values of ε the solutions increase fastly with time, but the negative coefficients of the non-linear terms act in order to stabilize the solution.

We are now in a position to examine the stability, with respect to small disturbances, of the fixed points corresponding to the equilibrium solutions of the set (68)–(69). There are nine fixed points about which the following comments can be made:

$$\text{I}: Y = Z = 0.$$

This situation corresponds to the purely conductive state: no motion is observed.

$$\text{II}_{a,b}: Y = 0, \quad Z = \pm(\varepsilon L / -R_1)^{1/2}.$$

Since $R_1 < 0$, this solution exists only for ε positive and represents a two-dimensional planform, i.e. rolls. The plus and minus signs (subcases a and b) describe rolls moving clockwise and counterclockwise, respectively.

$$\text{III}_{a,b}: Y = 2Z; \quad Z = [2(4R+P)]^{-1} \times [-\gamma \pm (\gamma^2 - 4\varepsilon L(4R+P))^{1/2}]$$

$$\text{IV}_{a,b}: Y = -2Z; \quad Z = [2(4R+P)]^{-1} \times [-\gamma \pm (\gamma^2 - 4\varepsilon L(4R+P))^{1/2}].$$

In the cases III and IV, solutions exist only for $\varepsilon < \varepsilon_c = \gamma^2 / 4L(4R+P)$, and represent hexagonal cells; a plus sign in Z corresponds to subcase a and a minus sign to subcase b. It is not necessary to consider solutions III and IV separately, because the system (68)–(69) is invariant under the transformation $Y \to -Y$.

$$\text{V}_{a,b}: Z = \frac{-\gamma}{P - R_1};$$

$$Y = \pm(-R)^{1/2} \left[\varepsilon L + \frac{R_1 \gamma^2}{(P - R_1)^2} \right]^{1/2}.$$

A solution exists for $\varepsilon > \varepsilon_1 = -R_1 \gamma^2 / L(P - R_1)^2$, a and b corresponds to the signs + and −, respectively. It follows from V that roll cells are obtained for $\varepsilon = \varepsilon_1$ and hexagonal cells for $\varepsilon = \varepsilon_2 = \gamma^2(4R + R_1)/(P - R_1)^2$; otherwise the pattern is hybrid.

It was observed that the cases a and b in solutions I–V are different in view of their stability analysis with respect to ε. The stability of the solutions I–V is summarized in Table 1. We observe that in the subcritical domain ($\varepsilon_c < \varepsilon < 0$), only hexagon cells are stable. For $0 < \varepsilon < \varepsilon_1$ only stable hexagons will be observed. For values of $\varepsilon > \varepsilon_1$, rolls as well as hexagons are stable. For sufficiently large values of $\varepsilon (\varepsilon > \varepsilon_2)$, rolls and hybrid cells are predicted. It is also

Table 1. The hierarchy of stable configurations as a function of parameter ε

$\varepsilon = \dfrac{Ma - Ma^c}{Ma^c}$	Stable configurations
$\varepsilon < \varepsilon_c$	Conductive state
$\varepsilon_c < \varepsilon < 0$	Conductive state, hexagons
$0 < \varepsilon < \varepsilon_1$	Hexagons
$\varepsilon_1 < \varepsilon < \varepsilon_2$	Rolls, hexagons
$\varepsilon > \varepsilon_2$	Rolls, hybrid cells

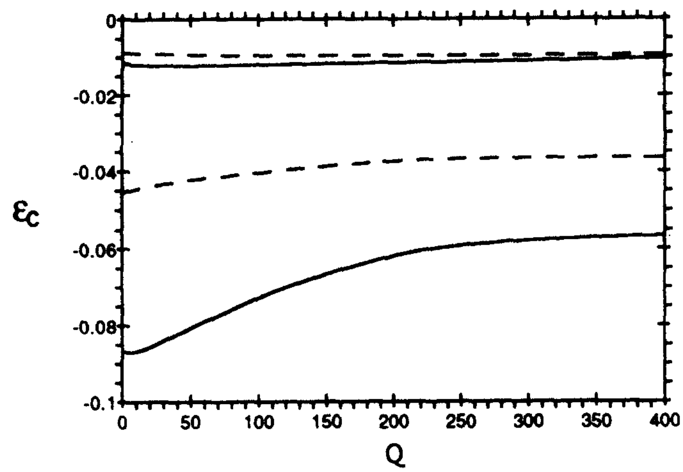

Fig. 2. Coefficient ε_c vs Chandrasekhar number, in the conducting (solid lines) and in insulating cases (dashed lines). The values of the Biot number are 1, 10 and in each case the lower curve corresponds to the largest Biot number.

checked that only hexagons for which the fluid is rising at the centre are stable.

The behaviour of the coefficients ε_c, ε_1 and ε_2, with respect to the Chandrasekhar Q and the Biot h number is reproduced in Figs. 2 and 3. Whatever the thermal property of the lower walls either insulating or conducting, the parameter ε may take negative values and this means that a region of subcritical instability is predicted; apart from a small region corresponding to low magnetic fields, the region of subcriticality decreases with increasing values of Q (see Fig. 2). Figures 3(a) and (b) show that at large values of Q, the values of ε_1 and ε_2 remain practically constant which means that above a critical value of Q, the application of a magnetic field leaves unmodified the structure of the planform. In contrast, for small magnetic field strengths and in the case of a conducting wall, ε_1 and ε_2 are seen to decrease and afterwards to relax slowly towards their asymptotic value. For insulating walls, ε_1 and ε_2 decrease for low values of h: at large values of the Biot number h, ε_1 and ε_2 increase with a maximum value for Q finite. It is also seen that ε_1 and ε_2 become larger and larger with increasing values of the heat transfer coefficient at the upper boundary.

4. CONCLUSIONS

A weakly non-linear analysis of Marangoni convection in a thin horizontal electrically conducting fluid subject to a vertical temperature gradient and a vertical external magnetic field is proposed. Several assumptions have been introduced, like absence of gravity effects, an infinite Prandtl number, no deformation of the upper surface of the layer which, in addition, is supposed to extend laterally to infinity, all the thermal, viscous and electromagnetic material coefficient, like viscosity, heat conductivity, resistivity are assumed to be constant. A semi-analytical solution is obtained based on the amplitude method developed by Scanlon and Segel [15]. We have also emphasized the differences obtained by assuming, respectively, that the lower surface of the layer is perfectly conducting and perfectly adiabatically insulated.

The main results that were obtained can be summarized as follows:

(1) In absence of magnetic fields, one recovers, as it should, earlier results obtained by other authors [16, 17].
(2) A region of subcritical instability is displayed and the area of this region is shown to decrease at sufficiently large values of the Chandrasekhar with the magnetic field.
(3) After that the instability has set in, convective cells taking the form of hexagons are predicted as far as the parameter $\varepsilon = (Ma - Ma^c)/Ma^c$ measuring the distance from the critical threshold Ma^c remains smaller than ε_1; it is shown that the value of ε_1 is rather sensitive to the strength of the magnetic field as long as Q remain small; by increasing Q, ε_1 remains practically constant.
(4) By still augmenting ε up to a value $\varepsilon = \varepsilon_2$, new patterns taking the form of rolls or hexagons are displayed. Like ε_1, this value ε_2 is mainly sensitive to Q for small Q-values.
(5) For ε larger than ε_1, rolls and hybrid cells are predicted. It is worth mentioning that the convective cells have qualitatively the same form as in absence of a magnetic field with the same hierarchy as exhibited by Table 1.

A last remark is in form. Since the present analysis is a weakly non-linear approach, the conclusions about the parameters ε_1 and ε_2, when they take values much larger than one, must be considered as only qualitatively valuable. A more general non-linear approach overcoming this difficulty is presently under progress.

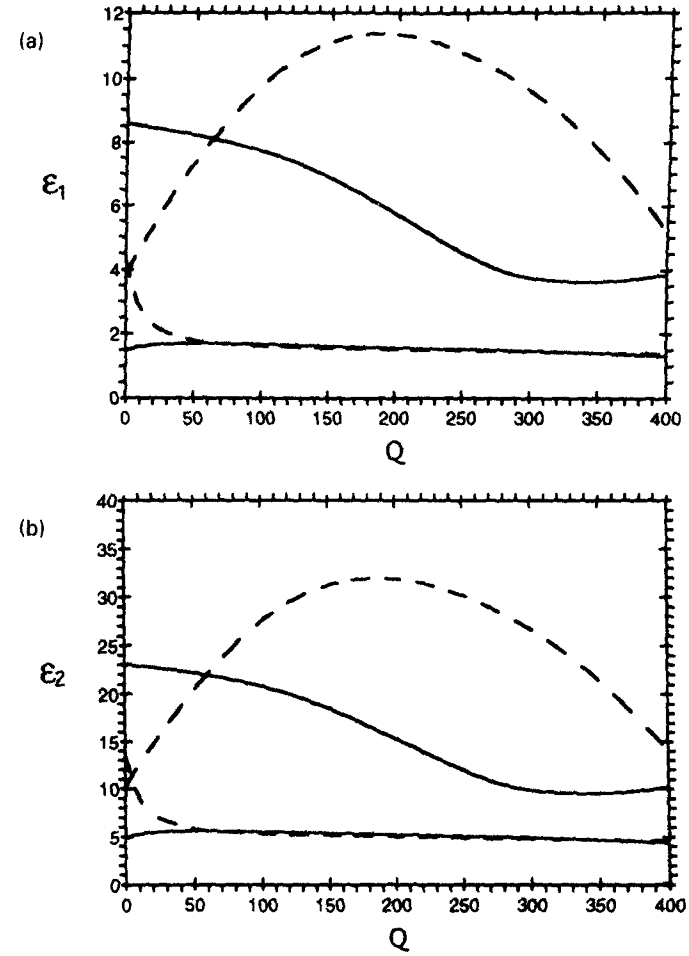

Fig. 3. (a) Coefficient ε_1 vs Chandrasekhar number, in the conducting (solid lines) and in insulating cases (dashed lines). The values of the Biot number are 1, 10 and in each case the lower curve corresponds to the smallest Biot number; (b) coefficient ε_2 vs Chandrasekhar number, in the conducting (solid lines) and in insulating cases (dashed lines). The values of the Biot number are 1, 10 and in each case the lower curve corresponds to the smallest Biot number.

Acknowledgement—This test presents results of the Belgian Programme on Interuniversity Poles of Attraction (PAI 4-06) initiated by the Belgian State, Prime Minister's Office, Science Policy Programming.

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APPENDIX A

The linear adjoint problem
The linear problem may be written in the form

$$[\bar{L}-Ma^c M]\mathbf{u} = 0. \qquad (A1)$$

Besides Marangoni boundary condition, the remaining ones are

$$z = 0: w = \bar{D}w = 0 \qquad (A2)$$

$$z = 1: w = \bar{D}T+hT = 0. \qquad (A3)$$

The adjoint operator is defined by

$$\langle \mathbf{u}^*, [\bar{L}-Ma^c M]\mathbf{u}\rangle = \langle \mathbf{u}, [\bar{L}^*, Ma^c M^*]\mathbf{u}^*\rangle \qquad (A4)$$

where $\mathbf{u}^* = (w^*, T^*, \bar{D}w^*_{(1)})$ is the adjoint eigenvector solution of

$$[\bar{L}^*-Ma^c M^*]\mathbf{u}^* = 0. \qquad (A5)$$

By integrating by parts the left-hand side of (A4) and using (A2) and (A3), one obtains

$$[\bar{L}^*-Ma^c M^*] = \begin{bmatrix} \nabla^4-Q\bar{D}^2 & 1 & 0 \\ 0 & \nabla^2 & 0 \\ 0 & (\bar{D}+h)_{z=1} & Ma^c\nabla_1^2 \end{bmatrix}$$

with the corresponding boundary conditions:

$$\text{at } z = 0: w^* = \bar{D}w^* = 0 \qquad (A6)$$

$$T^* = 0 \text{ (conducting case)}, \quad \bar{D}T^* = 0 \text{ (insulating case)} \qquad (A7)$$

$$\text{at } z = 1: w^* = \bar{D}^2 w^* = 0. \qquad (A8)$$

By analogy with the linear direct problem, let us write the solutions in the form of separable variables:

$$w^* = w^*(z)\phi(x,y,t), \quad T^* = T^*(z)\phi(x,y,t). \qquad (A9)$$

To determine the Fredholm condition, it is necessary to calculate $T^*(z)$ and $\bar{D}w^*(1)$, which are given by

$$T^*(z) = A\sinh az, \quad \bar{D}w^*(1) = A\frac{a\cosh a+h\sinh a}{a^2 Ma^c} \qquad (A10)$$

for the conducting case, for the insulating case, it is found that

$$T^*(z) = A\cosh az, \quad \bar{D}w^*(1) = A\frac{a\sinh a+h\cosh a}{a^2 Ma^c} \qquad (A11)$$

A is an arbitrary constant which is not rel calculations.

APPENDIX B

Second-order solution
The problem we are faced with is the resolution of equations (54)–(56) associated with the boundary conditions (57)–(60) by using expressions (61)–(64) for the functions ϕ_i $(i = 1,2,3,4)$. Since the functions ϕ and ϕ_2 contain two terms, one proportional to $\cos ay$ and the other to $\cos am_1x\cos am_2y$, the corresponding functions $K(z)$ and $T_2(z)$ will also consist in two terms. The functions $K(z)$ and $T_2(z)$ are written as the sum of the solutions $\tilde{T}(z,t)$ and $\tilde{\tilde{T}}(z,t)$ proportional to $\cos ay$ and $\cos am_1x\cos am_2y$, respectively, with $\tilde{T}$ and $\tilde{\tilde{T}}$ satisfying the equations:

$$(\bar{D}^2-a^2)\tilde{T} = T_1\dot{Z}+\frac{Y^2}{4}\left(T_1\frac{\bar{D}w_1}{2}+w_1\bar{D}T_1\right) \qquad (B1)$$

$$(\bar{D}^2-a^2)\tilde{\tilde{T}} = T_1\dot{Y}+YZ\left(T_1\frac{\bar{D}w_1}{2}+w_1\bar{D}T_1\right) \qquad (B2)$$

and the boundary equations (57)–(60). After solving the problem by the Green function procedure, one obtains, in the conducting case:

$$K(z) = \frac{-1}{a(a\cosh a+h\sinh a)}\left\{\int_0^z \sinh au(a\cosh a(1-z) + h\sinh a(1-z))T_1(u)\,du + \int_z^1 \sinh az(a\cosh a(1-u) + h\sinh a(1-u))T_1(u)\,du\right\} \qquad (B3)$$

and

$$T_2(z) = \frac{-1}{a(a\cosh a+h\sinh a)} \times\left\{\int_0^z \sinh au(a\cosh a(1-z)+h\sinh a(1-z)) \times F(u)\,du + \int_z^1 \sinh az(a\cosh a(1-u) + h\sinh a(1-u))F(u)\,du\right\} \qquad (B4)$$

with

$$F(u) = w_1(u)\bar{D}T_1(u)+\frac{T_1(u)\bar{D}w_1(u)}{2}.$$

For the insulating case, it suffices to substitute $\sinh au \sinh az$, $\sinh a$ and $\cosh a$ by $\cosh au$, $\cosh az$, $\cosh a$ and $\sinh a$ respectively. The functions $w_3(z)$ and $w_4(z)$ are solutions to

$$[(\bar{D}^2-ja^2)^2-Q\bar{D}^2]w_j(z) = 0, \quad (j = 3,4) \qquad (B5)$$

with the boundary conditions

$$z = 0: \quad w_j = \bar{D}w_j = 0 \qquad (B6)$$

$$z = 1: \quad w_j = 0. \qquad (B7)$$

They will be written as

$$w_j = A_j[\cosh\alpha_j z + B_j\sinh\alpha_j z+C_j\cosh\beta_j z+D_j\sinh\beta_j z] \qquad (B8)$$

where B_j, C_j and D_j are obtained from the boundary conditions (B6) and (B7): they have the same form as B, C, and D in equation (42) by replacing α by α_j, β by β_j and Δ by $\Delta_j = Q+4ja^2$. The constants A_j $(j = 3, 4)$ have to be determined together with the functions $T_3(z)$ and $T_4(z)$, the latter satisfy the equations:

$$(\bar{D}^2 - ja^2)T_j(z) = -w_j(z) + w_1 \bar{D}T - \delta T_1 \bar{D}w_1 \equiv F(z, A_j) \tag{B9}$$

wherein $\delta = 1/2$ if $j = 3$ and $\delta = 1$ if $j = 4$, with at $z = 1$

$$\bar{D}^2 w_j + ja^2 Ma^c T_j(z) = 0. \tag{B10}$$

Applying the Green function method, the above set may be written, for the conducting case, as

$$\begin{aligned} T_j(z) = \int_0^z &- \frac{\sinh a\sqrt{j}u}{a\sqrt{j}} \\ &\times \frac{(a\sqrt{j}\cosh a\sqrt{j}(1-z) + h\sinh a\sqrt{j}(1-z))}{a\sqrt{j}\cosh a\sqrt{j} + h\sinh a\sqrt{j}} \\ &\times F(u, A_j)\,\mathrm{d}u - \int_z^1 \frac{\sinh a\sqrt{j}z}{a\sqrt{j}} \\ &\times \frac{(a\sqrt{j}\cosh a\sqrt{j}(1-u) + h\sinh a\sqrt{j}(1-u))}{a\sqrt{j}\cosh a\sqrt{j} + h\sinh a\sqrt{j}} \\ &\times F(u, A_j)\,\mathrm{d}u. \end{aligned} \tag{B11}$$

After making use of the boundary condition (B10), the constants A_j are given by

$$A_j = \frac{ja^2 Ma^c \int_0^1 \sinh(a\sqrt{j}u) F_j(u)\,\mathrm{d}u}{ja^2 Ma^c \int_0^1 \sinh(a\sqrt{j}u) w_j'(u)\,\mathrm{d}u + (a\sqrt{j}\cosh a\sqrt{j} + h\sinh a\sqrt{j})\bar{D}^2 w_j'(1)} \tag{B12}$$

with

$$w_j'(u) = (\cosh \alpha_j u + B_j \sinh \alpha_j u + C_j \cosh \beta_j u + D_j \sinh \beta_j u) \tag{B13}$$

$$F_j(u) = w_1(u)\bar{D}T_1(u) - \delta T_1(u)\bar{D}w_1(u) - w_j'(u). \tag{B14}$$

The corresponding expressions for the insulating case are obtained by changing $\sinh a\sqrt{j}u$, $\sinh a\sqrt{j}z$, $\sinh a\sqrt{j}$, $\cosh a\sqrt{j}$ in $\cosh a\sqrt{j}u$, $\cosh a\sqrt{j}z$, $\cosh a\sqrt{j}$, $\sinh a\sqrt{j}$, respectively, and for $F_j(u)$ given by (B14) by replacing $T_1(u)$ by its expression in the insulating case (41). The function $T_0(z)$ satisfies the following equation and boundary conditions:

$$\bar{D}^2 T_0 = T_1 \bar{D}w_1 + w_1 \bar{D}T_1 = G_0(z) \tag{B15}$$

$$\text{at } z = 0: T_0 = 0 \text{ (conducting case)}, \tag{B16}$$

$$\bar{D}T_0 = 0 \text{ (insulating case)}$$

$$\text{at } z = 1: \bar{D}T_0 + hT_0 = 0. \tag{B17}$$

By applying the Green function method, it is found that in the conducting case:

$$T_0(z) = \int_0^z \left(z\frac{h}{1+h} - 1\right) u G_0(u)\,\mathrm{d}u + \int_z^1 z\left(u\frac{h}{1+h} - 1\right) G_0(u)\,\mathrm{d}u \tag{B18}$$

and in the insulating case

$$T_0(z) = \int_0^z \left(z - \frac{1+h}{h}\right) G_0(u)\,\mathrm{d}u + \int_z^1 \left(u - \frac{1+h}{h}\right) G_0(u)\,\mathrm{d}u. \tag{B19}$$